Available at
www.ElsevierMathematics.com
POWERED BY SCIENCE @ DIRECT®

ELSEVIER

Journal of Computational and Applied Mathematics 161 (2003) 161–177

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

www.elsevier.com/locate/cam

Integral equation methods for scattering from an impedance crack

Rainer Kress*, Kuo-Ming Lee

Institut für Numerische und Angewandte Mathematik, Universität Göttingen, Lotzestr. 16-18, 37083 Göttingen, Germany

Received 28 February 2003; received in revised form 16 June 2003

Abstract

For the scattering problem for time-harmonic waves from an impedance crack in two dimensions, we give a uniqueness and existence analysis via a combined single- and double-layer potential approach in a Hölder space setting leading to a system of integral equations that contains a hypersingular operator. For its numerical solution we describe a fully discrete collocation method based on trigonometric interpolation and interpolatory quadrature rules including a convergence analysis and numerical examples.

Keywords: Helmholtz equation; Impedance crack; Integral equation; Collocation method

1. Introduction

Beginning with [7] in a series of papers [3,9,11] the direct and, to some extent, the inverse scattering problem for time-harmonic acoustic, electromagnetic and elastic waves from a crack in two dimensions has been considered for Dirichlet and Neumann boundary conditions by an integral equation approach in classical Hölder space settings. All these papers give a full treatment of the corresponding problems including the existence and uniqueness analysis of the resulting integral equations and their numerical solution via fully discrete collocation methods based on trigonometric interpolation. One of the key ingredients is the use of the cosine substitution introduced in [12,15] for related potential theoretic problems that, in principle, reduces the integral equation over a crack to an integral equation over a closed boundary curve.

* Corresponding author. Tel.: +49551394511; fax: +49551393944.
E-mail address: kress@math.uni-goettingen.de (R. Kress).

doi:10.1016/S0377-0427(03)00586-7

In this paper, we extend these methods to the case of the impedance boundary condition for acoustic and electromagnetic waves. Whereas for the Dirichlet and Neumann boundary condition a single- or a double-layer potential suffices for establishing existence, for the impedance boundary condition both potentials are required in the analysis. This leads to a system of two integral equations for two density functions that, as for the Neumann boundary condition, contains the hypersingular operator of the normal derivative of the double-layer potential. Furthermore, as opposed to the Dirichlet and Neumann problem the cosine substitution only leads to partial success: although it enables an elegant existence analysis, in general, it only leads to less favorable convergence rates in the numerical solution. This resembles a different structure of the singularities of the solution to the impedance problem at the end points of the crack as compared to the Dirichlet or Neumann boundary condition. For the latter, the solution u in the vicinity of the end points behaves like a product $u = \sqrt{\rho}v$ where ρ denotes the distance to the end point and v is a function that, roughly speaking, is as smooth as the crack and the boundary data. This explains the rapid convergence of the methods based on the cosine transformation since this substitution takes complete care of the end point singularities. For the impedance boundary condition (with the same impedance on both sides of the crack) the solution u still behaves like $u = \sqrt{\rho}v$. However, now the function v contains further singularities of square root and logarithmic type (see [5,13]). As further research, in the spirit of [6] it is intended to use graded meshes via sigmoidal substitutions to satisfactory deal with these additional singularities and speed up the convergence of the numerical method.

A somewhat similar approach to scattering from an impedance crack, with Dirichlet condition on one side of the crack, via a combination of a single- and a double-layer potential was considered in [11]. However, the analysis in [1] is performed within a Sobolev space setting and a weak solution concept and no numerical method is presented. The main purpose of [1] actually is the solution of the inverse scattering problem and for the numerical examples the authors state that they tested their method only for the case of the Dirichlet boundary condition since a forward solver for the impedance boundary value problem for cracks as needed to produce synthetic data was not available to them. The present paper now may serve as a remedy with this respect.

The plan of the paper is as follows. In Section 2, we will introduce our formulation of the impedance crack problem and prove uniqueness of a solution. Then in Section 3, we proceed by establishing the existence of a solution via integral equation methods and Section 4 is devoted to describing the numerical solution of the integral equation. Finally, we conclude with some numerical examples in Section 5.

2. Uniqueness

Assume that $\Gamma \subset \mathbb{R}^2$ is an infinitely differentiable arc, i.e.,

$$\Gamma = \{z(t) : t \in [-1, 1]\},$$

where $z : [-1, 1] \to \mathbb{R}^2$ is an injective and infinitely differentiable function. By z_1, z_{-1} we denote the two end points $z_1 := z(1)$ and $z_{-1} := z(-1)$ of Γ and set $\Gamma_0 := \Gamma \setminus \{z_{-1}, z_1\}$. Assuming an orientation for Γ from z_{-1} to z_1, by Γ_+ and Γ_- we denote the left- and right-hand sides of Γ, respectively, and by ν the unit normal vector to Γ directed towards Γ_+.

The mathematical modelling for scattering of time-harmonic acoustic or electromagnetic waves from thin infinitely long cylindrical coated objects, i.e., from coated cracks, leads to the following impedance boundary value problem for the Helmholtz equation in the exterior of Γ: Find a solution $u \in C^2(\mathbb{R}^2 \setminus \Gamma) \cap C(\mathbb{R}^2 \setminus \Gamma_0)$ of the Helmholtz equation

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^2 \setminus \Gamma$$

with wave number $k > 0$ satisfying the impedance boundary condition

$$\frac{\partial u_\pm}{\partial \nu} \pm \mathrm{i}k\lambda u_\pm = 0 \quad \text{on } \Gamma_0,$$

where the limits

$$u_\pm(x) := \lim_{h \to 0} u(x \pm h\nu(x)), \quad x \in \Gamma, \tag{2.1}$$

and

$$\frac{\partial u_\pm(x)}{\partial \nu} := \lim_{h \to +0} \nu(x) \operatorname{grad} u(x \pm h\nu(x)), \quad x \in \Gamma_0, \tag{2.2}$$

are required to exist in the sense of local uniform convergence. Further, $\lambda \in C^{0,\alpha}(\Gamma)$ is a given Hölder continuous function with

$$\operatorname{Re} \lambda \geqslant 0 \tag{2.3}$$

representing the impedance of the crack Γ. We note that, in principle, the following analysis can be extended to the case of different impedance functions on Γ_+ and Γ_-. The total field u is decomposed $u = u^{\mathrm{i}} + u^{\mathrm{s}}$ into the given incident field u^{i}, for example a plane wave $u^{\mathrm{i}}(x) = \mathrm{e}^{\mathrm{i}kx\cdot d}$ with $|d| = 1$, and the unknown scattered field u^{s} which is required to satisfy the Sommerfeld radiation condition. After renaming the unknown function, this scattering problem is a special case of the following *impedance crack problem*: Given two functions $f_\pm \in C^{0,\alpha}(\Gamma)$, find a solution $u \in C^2(\mathbb{R}^2 \setminus \Gamma) \cap C(\mathbb{R}^2 \setminus \Gamma_0)$ to the Helmholtz equation

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^2 \setminus \Gamma \tag{2.4}$$

which satisfies the boundary condition

$$\frac{\partial u_\pm}{\partial \nu} \pm \mathrm{i}k\lambda u_\pm = f_\pm \quad \text{on } \Gamma_0 \tag{2.5}$$

and the Sommerfeld radiation condition

$$\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial u}{\partial r} - \mathrm{i}ku \right) = 0, \quad r = |x|, \tag{2.6}$$

uniformly in all directions $\hat{x} = x/|x|$.

Uniqueness of a solution relies on Rellich's lemma and the application of Green's theorem with the end points of Γ requiring special consideration as indicated in the following lemma.

Lemma 2.1. *Let u be a solution to the impedance crack problem with homogeneous boundary condition and denote by B_R the open disk of radius R centered at the origin with outer unit normal*

ν to the boundary ∂B_R. Then for sufficiently large R we have that $\operatorname{grad} u \in L^2(B_R)$ *and*

$$\int_{B_R} \{|\operatorname{grad} u|^2 - k^2|u|^2\}\,\mathrm{d}x = \int_{\partial B_R} u\,\frac{\partial \bar u}{\partial \nu}\,\mathrm{d}s - \mathrm{i}k \int_\Gamma \bar\lambda\{|u_+|^2 + |u_-|^2\}\,\mathrm{d}s. \tag{2.7}$$

Proof. For $r > 0$ and $j = -1, 1$ we introduce the disks $B_{r,j}$ of radius r centered at z_j with the exterior unit normal vector ν to the boundary $\partial B_{r,j}$. Modifying Mönch's [11] proof for the Neumann boundary condition, we choose a function $v \in C^2(\mathbb{R}^2)$ such that $v(x) = u(z_j)$ for all $z \in B_{r_0,j}$, $j = -1, 1$, and some r_0 with $0 < 2r_0 < |z_1 - z_{-1}|$. We consider $w := u - v$ and after choosing an odd real-valued function $\psi \in C^1(\mathbb{R})$ such that $\psi(t) = 0$ for $0 \leqslant t \leqslant 1$, $\psi(t) = t$ for $t \geqslant 2$ and $\psi'(t) \geqslant 0$ for all t, we set $w_n := \psi(n \operatorname{Re} w)/n + \mathrm{i}\psi(n \operatorname{Im} w)/n$. Note that w_n vanishes in a neighborhood of the end points z_1 and z_{-1}. Therefore, we can apply Green's theorem to w_n and $\bar u$, use the homogeneous impedance condition and pass to the limit $n \to \infty$ with the aid of Fatou's lemma and Lebesgue's dominated convergence theorem (see also [4, Lemma 3.8] to obtain that $\operatorname{grad} u \in L^2(B_R)$ and

$$\int_{B_R} \{|\operatorname{grad} u|^2 - \operatorname{grad} v \operatorname{grad} \bar u - k^2 w \bar u\}\,\mathrm{d}x$$
$$= \int_{\partial B_R} w\,\frac{\partial \bar u}{\partial \nu}\,\mathrm{d}s - \mathrm{i}k \int_\Gamma \bar\lambda\{w_+ \bar u_+ + w_- \bar u_-\}\,\mathrm{d}s. \tag{2.8}$$

Since v is constant in the disks $B_{r_0,1}$ and $B_{r_0,-1}$, again from Green's theorem we have that

$$\int_{B_R} \{\operatorname{grad} v \operatorname{grad} \bar u - k^2 v \bar u\}\,\mathrm{d}x = \int_{\partial B_R} v\,\frac{\partial \bar u}{\partial \nu}\,\mathrm{d}s$$
$$-I_1(r) - I_{-1}(r) - \mathrm{i}k \int_{\Gamma \setminus (B_{r,1} \cup B_{r,-1})} \bar\lambda v \{\bar u_+ + \bar u_-\}\,\mathrm{d}s \tag{2.9}$$

for all $0 < r < r_0$, where for $j = -1, 1$ we have set

$$I_j(r) := \int_{\partial B_{r,j}} v\,\frac{\partial \bar u}{\partial \nu}\,\mathrm{d}s.$$

Using the Cauchy–Schwarz inequality we can estimate

$$|I_j(r)|^2 \leqslant C_j r \int_{\partial B_{r,j}} \left|\frac{\partial u}{\partial \nu}\right|^2 \mathrm{d}s \leqslant C_j r \int_{\partial B_{r,j}} |\operatorname{grad} u|^2\,\mathrm{d}s,$$

where $C_j := 2\pi |u(z_j)|^2$ and, consequently,

$$\int_0^{r_0} \frac{1}{r}\,|I_j(r)|^2\,\mathrm{d}r \leqslant C_j \int_{B_{r_0,j}} |\operatorname{grad} u|^2\,\mathrm{d}x \leqslant C_j \|\operatorname{grad} u\|^2_{L^2(B_R)}.$$

Therefore $\liminf_{r\to 0} I_j(r) = 0$ and the statement of the lemma follows by letting r tend to zero in (2.9) and combining the result with (2.8). □

Theorem 2.2. *The impedance crack problem has at most one solution.*

Proof. In view of assumption (2.3), taking the imaginary part of (2.7) yields

$$\operatorname{Im}\int_{\partial B_R} u\,\frac{\partial\bar{u}}{\partial\nu}\,\mathrm{d}s \geqslant 0$$

and the assertion follows from [4, Theorem 2.12]. □

3. Existence

We will establish existence of a solution to the impedance crack problem by a combined single- and double-layer potential approach. Although we do not explicitly include any edge condition at the two end points z_1 and z_{-1} in the formulation of the impedance crack problem, in the existence analysis we anticipate the singular behavior of the solution through the following regularity assumptions for the densities of the potentials. For $0<\alpha<1$ we introduce a space of locally Hölder continuously differentiable functions on Γ by

$$C^{1,\alpha}_{0,\mathrm{loc}}(\Gamma) := C^{1,\alpha}_{\mathrm{loc}}(\Gamma_0)\cap\{\varphi\in C(\Gamma):\varphi(z_{-1})=\varphi(z_1)=0,\varphi'\in L^1(\Gamma)\},$$

where the prime indicates differentiation with respect to arc length. We seek the solution in the form

$$u(x)=\int_\Gamma \Phi(x,y)\varphi_1(y)\,\mathrm{d}s(y)+\int_\Gamma \frac{\partial\Phi(x,y)}{\partial\nu(y)}\,\varphi_2(y)\,\mathrm{d}s(y),\quad x\in\mathbb{R}^2\backslash\Gamma, \tag{3.1}$$

with densities $\varphi_1\in C(\Gamma)$ and $\varphi_2\in C^{1,\alpha}_{0,\mathrm{loc}}(\Gamma)$, where the fundamental solution to the Helmholtz equation in two dimensions is given by

$$\Phi(x,y) := \frac{\mathrm{i}}{4}H_0^{(1)}(k|x-y|),\quad x\neq y,$$

in terms of the Hankel function $H_0^{(1)}$ of order zero and of the first kind. Clearly, (3.1) satisfies the radiation condition. From the jump relations (see [4]) it can be deduced that our assumptions on the densities ensure that $u\in C^2(\mathbb{R}^2\backslash\Gamma)\cap C(\mathbb{R}^2\backslash\Gamma_0)$ and that the limits (2.1) and (2.2) exist. After rewriting the two boundary conditions (2.5) in the equivalent form of their difference and their sum, the jump relations also imply that (3.1) satisfies the boundary condition provided the densities φ_1 and φ_2 solve the system of integral equations

$$\begin{aligned}&-\varphi_1+\mathrm{i}k\lambda S\varphi_1+\mathrm{i}k\lambda K\varphi_2=f_1,\\ &K'\varphi_1+T\varphi_2+\mathrm{i}k\lambda\varphi_2=f_2\end{aligned} \tag{3.2}$$

on Γ_0. Here, for convenience we have abbreviated $f_1 := f_+ - f_-$ and $f_2 := f_+ + f_-$ and the operators are defined by

$$(S\varphi)(x) := 2\int_\Gamma \Phi(x,y)\varphi(y)\,\mathrm{d}s(y),$$

$$(K\varphi)(x) := 2\int_\Gamma \frac{\partial\Phi(x,y)}{\partial\nu(y)}\,\varphi(y)\,\mathrm{d}s(y),$$

$$(K'\varphi)(x) := 2\int_\Gamma \frac{\partial \Phi(x,y)}{\partial \nu(x)}\,\varphi(y)\,\mathrm{d}s(y),$$

$$(T\varphi)(x) := 2\,\frac{\partial}{\partial \nu(x)}\int_\Gamma \frac{\partial \Phi(x,y)}{\partial \nu(y)}\,\varphi(y)\,\mathrm{d}s(y)$$

for $x \in \Gamma_0$. For the hypersingular operator T we have Maue's identity

$$T\varphi = \frac{\mathrm{d}}{\mathrm{d}s}\,S\,\frac{\mathrm{d}\varphi}{\mathrm{d}s} + k^2 \nu \cdot S(\nu\varphi) \quad \text{on } \Gamma_0 \tag{3.3}$$

for densities $\varphi \in C^{1,\alpha}_{0,\mathrm{loc}}(\Gamma)$. As in the case of close boundary curves (see [10, Theorem 7.29; 11]) it can be obtained by partial integration using $\varphi(z_1) = \varphi(z_{-1}) = 0$.

We note that in view of Theorem 2.2 the jump relations also imply that system (3.2) has at most one solution, i.e., we can state the following theorem.

Theorem 3.1. *The system of integral equations* (3.2) *has at most one solution* $\varphi_1 \in C(\Gamma)$ *and* $\varphi_2 \in C^{1,\alpha}_{0,\mathrm{loc}}(\Gamma)$.

To proceed with the existence analysis and for the numerical solution we parameterize the integral equations (3.2). Following [4,8,11], for $-1 < t < 1$ we can write

$$(S\varphi)(z(t)) = \int_{-1}^{1} L_1(t,\tau)\varphi(z(\tau))\,\mathrm{d}\tau,$$

$$(K\varphi)(z(t)) = \int_{-1}^{1} L_2(t,\tau)\varphi(z(\tau))\,\mathrm{d}\tau,$$

$$(K'\varphi)(z(t)) = \frac{1}{|z'(t)|}\int_{-1}^{1} L_3(t,\tau)\varphi(z(\tau))\,\mathrm{d}\tau,$$

$$(T\varphi)(z(t)) = \frac{1}{|z'(t)|}\int_{-1}^{1}\left\{\frac{1}{2\pi}\,\frac{1}{\tau - t}\,\frac{\mathrm{d}}{\mathrm{d}\tau}\,\varphi(z(\tau)) + L_4(t,\tau)\varphi(z(\tau))\right\}\mathrm{d}\tau \tag{3.4}$$

with the kernels

$$L_1(t,\tau) := \frac{\mathrm{i}}{2} H_0^{(1)}(k|z(t)-z(\tau)|)|z'(\tau)|,$$

$$L_2(t,\tau) := \frac{\mathrm{i}k}{2} H_1^{(1)}(k|z(t)-z(\tau)|)\frac{\{z(t)-z(\tau)\}(z_2'(\tau), -z_1'(\tau))}{|z(t)-z(\tau)|},$$

$$L_3(t,\tau) := |z'(\tau)|\,L_2(\tau,t)$$

and

$$L_4(t,\tau) := -\frac{\mathrm{i}}{2}\,\frac{z'(t)\{z(t)-z(\tau)\}z'(\tau)\{z(t)-z(\tau)\}}{|z(t)-z(\tau)|^2}$$
$$\times\left\{k^2H_0^{(1)}(k|z(t)-z(\tau)|)-\frac{2kH_1^{(1)}(k|z(t)-z(\tau)|)}{|z(t)-z(\tau)|}\right\}$$
$$-\frac{\mathrm{i}k}{2}\,\frac{z'(t)z'(\tau)}{|z(t)-z(\tau)|}\,H_1^{(1)}(k|z(t)-z(\tau)|)$$
$$-\frac{1}{\pi}\,\frac{1}{(\tau-t)^2}+\frac{\mathrm{i}k^2}{2}\,H_0^{(1)}(k|z(t)-z(\tau)|)z'(t)z'(\tau),$$

where $H_1^{(1)}$ is the Hankel function of order one. (For the hypersingular operator T Maue's identity (3.3) is incorporated via partial integrations.) To acknowledge the logarithmic singularities of these kernels at $t=\tau$ we split them into

$$L_j(t,\tau) = M_j(t,\tau)\ln 2|t-\tau| + N_j(t,\tau),$$

where

$$M_1(t,\tau) := -\frac{1}{\pi}\,J_0(k|z(t)-z(\tau)|)|z'(\tau)|,$$

$$M_2(t,\tau) := -\frac{k}{\pi}\,J_1(k|z(t)-z(\tau)|)\,\frac{\{z(t)-z(\tau)\}(z_2'(\tau),-z_1'(\tau))}{|z(t)-z(\tau)|},$$

$$M_4(t,\tau) := \frac{1}{\pi}\,\frac{z'(t)\{z(t)-z(\tau)\}z'(\tau)\{z(t)-z(\tau)\}}{|z(t)-z(\tau)|^2}$$
$$\times\left\{k^2J_0(k|z(t)-z(\tau)|)-\frac{2kJ_1(k|z(t)-z(\tau)|)}{|z(t)-z(\tau)|}\right\}$$
$$+\frac{k}{\pi}\,\frac{z'(t)\cdot z'(\tau)}{|z(t)-z(\tau)|}\,J_1(k|z(t)-z(\tau)|)-\frac{k^2}{\pi}\,J_0(k|z(t)-z(\tau)|)z'(t)z'(\tau).$$

The functions M_j and N_j can be shown to be infinitely differentiable for $j=1,2,3,4$ with diagonal values

$$M_4(t,t) = -\frac{k^2|z'(t)|^2}{2\pi}$$

and

$$N_1(t,t) := \left(\frac{\mathrm{i}}{2}-\frac{C}{\pi}-\frac{1}{\pi}\ln\frac{k|z'(t)|}{4}\right)|z'(t)|,$$

$$N_2(t,t) := \frac{z_2'(t)z_1''(t) - z_1'(t)z_2''(t)}{2\pi|z'(t)|^2},$$

$$N_4(t,t) := \left(1 + \pi\mathrm{i} - 2C - 2\ln\frac{k|z'(t)|}{4}\right)\frac{k^2|z'(t)|^2}{4\pi}$$
$$-\frac{[z'(t)\cdot z''(t)]^2}{2\pi|z'(t)|^4} + \frac{|z''(t)|^2}{4\pi|z'(t)|^2} + \frac{z'(t)\cdot z'''(t)}{6\pi|z'(t)|^2},$$

where $C = 0.57721\ldots$ is Euler's constant.

By first parametrizing $x=z(t)$ and $y=z(\tau)$, then substituting $t=\cos s$ and $\tau=\cos\sigma$ and multiplying the second equation by $\sin s|z'(\cos s)|$ we transform (3.2) into the system

$$\begin{pmatrix} -I & 0 \\ 0 & U \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} + \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}+B \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \tag{3.5}$$

where we have set

$$\psi_1(s) := \varphi_1(z(\cos s)), \qquad \psi_2(s) := \varphi_2(z(\cos s)) \tag{3.6}$$

and

$$g_1(s) := f_1(z(\cos s)), \qquad g_2(s) := \sin s|z'(\cos s)|f_2(z(\cos s)).$$

The operators occurring in (3.5) are the integral operators

$$(U\psi)(s) := \frac{1}{\pi}\int_0^\pi \frac{\sin s}{\cos s - \cos\sigma}\,\psi'(\sigma)\,\mathrm{d}\sigma,$$

$$(A_{11}\psi)(s) := \mathrm{i}k\lambda(z(\cos s))\int_0^\pi L_1(\cos s,\cos\sigma)\sin\sigma\psi(\sigma)\,\mathrm{d}\sigma,$$

$$(A_{12}\psi)(s) := \mathrm{i}k\lambda(z(\cos s))\int_0^\pi L_2(\cos s,\cos\sigma)\sin\sigma\psi(\sigma)\,\mathrm{d}\sigma,$$

$$(A_{21}\psi)(s) := \sin s\int_0^\pi L_3(\cos s,\cos\sigma)\sin\sigma\psi(\sigma)\,\mathrm{d}\sigma,$$

$$(A_{22}\psi)(s) := \sin s\int_0^\pi L_4(\cos s,\cos\sigma)\sin\sigma\psi(\sigma)\,\mathrm{d}\sigma$$

and the multiplication operator

$$(B\psi)(s) := \mathrm{i}k\sin s|z'(\cos s)|\lambda(z(\cos s))\psi(s).$$

We introduce

$$C_0^{j,\alpha}[0,\pi] := \{\psi\in C^{j,\alpha}[0,\pi] : \psi(0)=\psi(\pi)=0\}$$

for $j = 0,1$ and proceed by establishing unique solvability of the parametrized integral equation (3.5) in $C^{0,\alpha}[0,\pi]\times C_0^{1,\alpha}[0,\pi]$. We note that for $\psi_1\in C^{0,\alpha}[0,\pi]$ and $\psi_2\in C_0^{1,\alpha}[0,\pi]$, via (3.6), the corresponding densities satisfy $\varphi_1 \in C(\Gamma)$ and $\varphi_2 \in C^{1,\alpha}_{0,\mathrm{loc}}(\Gamma)$. Therefore, Theorem 3.1 implies the uniqueness of a solution to (3.5) in $C^{0,\alpha}[0,\pi] \times C^{1,\alpha}_0[0,\pi]$.

By using the identity

$$\frac{\sin s}{\cos s - \cos\sigma} = \frac{1}{2}\cot\frac{\sigma - s}{2} - \frac{1}{2}\cot\frac{\sigma + s}{2},$$

we can write

$$\frac{1}{\pi}\int_0^\pi \frac{\sin s}{\cos s - \cos\sigma}\,\psi'(\sigma)\,\mathrm{d}\sigma = \frac{1}{2\pi}\int_0^{2\pi} \cot\frac{\sigma - s}{2}\,\chi'(\sigma)\,\mathrm{d}\sigma, \tag{3.7}$$

where χ denotes the odd and 2π periodic extension of ψ onto all of $\mathbb{R}$. We observe that for $\psi \in C^{1,\alpha}_0[0,\pi]$ the extension χ belongs to $C^{1,\alpha}(\mathbb{R})$. Hence from the standard mapping properties of the operator with Hilbert kernel in Hölder spaces (see [10]) it can be concluded that the operator $U : C^{1,\alpha}_0[0,\pi] \to C^{0,\alpha}_0[0,\pi]$ is bounded and has a bounded inverse (see also [3, Lemma 4]). This implies that the operator

$$\begin{pmatrix} -I & 0 \\ 0 & U \end{pmatrix} : C^{0,\alpha}[0,\pi] \times C^{1,\alpha}_0[0,\pi] \to C^{0,\alpha}[0,\pi] \times C^{0,\alpha}_0[0,\pi]$$

is bounded and has a bounded inverse. Because of the smoothness of the kernels of the integral operators and the compactness of the imbedding from $C^{1,\alpha}_0[0,\pi]$ into $C^{0,\alpha}_0[0,\pi]$ the operator

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} + B \end{pmatrix} : C^{0,\alpha}[0,\pi] \times C^{1,\alpha}_0[0,\pi] \to C^{0,\alpha}[0,\pi] \times C^{0,\alpha}_0[0,\pi]$$

can be seen to be compact. (For the integral operators with logarithmic kernels we refer to Lemma 4.1 in [8].) Hence, the following theorem is a consequence of the Riesz theory for compact operators.

Theorem 3.2. *The hypersingular integral equation* (3.5) *has a unique solution in* $C^{0,\alpha}[0,\pi] \times C^{1,\alpha}_0[0,\pi]$.

This finally implies existence (and uniqueness) of a solution to the impedance crack problem.

As opposed to the limiting case $\lambda = 0$ (see [3,11]), where the corresponding equation allows an odd extension to all of $\mathbb{R}$, in Eq. (3.5) some terms have an odd extension whereas others have an even extension, for example if λ is constant. Hence, in agreement with the general results on the end point singularities (see [5,13]), we cannot improve on the regularity of the solution to (3.5). This will effect the speed of convergence for the numerical solution method of the next section.

4. Numerical solution

For the numerical solution of the system of integral equations (3.5) we describe a collocation method based on trigonometric interpolation that is made fully discrete via appropriate quadrature approximations of the integral operators. For $n \geqslant 2$, we define $T_{n,1}$ as the space of cosine polynomials of degree less than or equal to n, i.e., the even trigonometric polynomials of degree less than or equal to n, and $T_{n,2}$ as the space of sine polynomials of degree less than or equal to $n-1$, i.e., the odd trigonometric polynomials of degree less than or equal to $n-1$. As collocation points we introduce the equidistant knots

$$s_j := \frac{j\pi}{n}, \quad j = 0,\dots,n$$

and denote by $P_{n,1}$ and $P_{n,2}$ the projection operators onto $T_{n,1}$ and $T_{n,2}$ that are uniquely defined via interpolation at $s_0,\dots,s_n$ and $s_1,\dots,s_{n-1}$, respectively. (For notational brevity, here and in the sequel, we do not indicate the dependence of the nodal points and of the quadrature weights on n.) We note that

$$(P_{n,1}\psi)(s) = \frac{2}{n}\sum_{j=0}^{n}\delta_j\psi(s_j)\sum_{k=0}^{n}\delta_k\cos ks_j\cos ks \tag{4.1}$$

and

$$(P_{n,2}\psi)(s) = \frac{2}{n}\sum_{j=1}^{n-1}\psi(s_j)\sum_{k=1}^{n-1}\sin ks_j\sin ks. \tag{4.2}$$

Here, for convenience, we have set $\delta_j = 1$ for $j = 1,\dots,n-1$ and $\delta_0 = \delta_n = \frac{1}{2}$. We will approximate ψ_1 and ψ_2 by trigonometric polynomials $\psi_{n,1}\in T_{n,1}$ and $\psi_{n,2}\in T_{n,2}$. Transformed back on the interval $[-1,1]$, this corresponds to approximating the density φ_1 by a linear combination of Chebyshev polynomials of the first kind of degree less than or equal to n and the density φ_2 by a product of $\sqrt{1-t^2}$ and a linear combination of Chebyshev polynomials of the second kind and of degree less than or equal to $n-2$. Therefore to some extent, the following approach resembles a method in [2] and differs from it through the choice of collocation points.

In addition to collocating Eq. (3.5) via the projection operator

$$P_n := \begin{pmatrix} P_{n,1} & 0 \\ 0 & P_{n,2}\end{pmatrix},$$

we must approximate the integral operators via numerical quadratures to arrive at a fully discrete method. For this we need quadrature rules for the integrals

$$\int_0^\pi \ln 2|\cos s - \cos\sigma|\psi(\sigma)\,\mathrm{d}\sigma \quad \text{and} \quad \int_0^\pi \psi(\sigma)\,\mathrm{d}\sigma$$

for even and odd 2π periodic functions and obtain them via approximating $\psi \approx P_{n,1}\psi$ and $\psi \approx P_{n,2}\psi$, respectively. By using the identity

$$\ln 4(\cos s - \cos\sigma)^2 = \ln 4\sin^2\frac{s-\sigma}{2} + \ln 4\sin^2\frac{s+\sigma}{2},$$

we transform

$$\int_0^\pi \ln 4(\cos s-\cos\sigma)^2\psi(\sigma)\,\mathrm{d}\sigma = \int_0^\pi\left\{\ln 4\sin^2\frac{s-\sigma}{2} + \ln 4\sin^2\frac{s+\sigma}{2}\right\}\psi(\sigma)\,\mathrm{d}\sigma.$$

From this and [10, Lemma 8.21] we obtain that

$$\int_0^\pi \ln 2|\cos s - \cos\sigma|\cos k\sigma\, \mathrm{d}\sigma = \begin{cases} 0, & k=0, \\ -\dfrac{\pi}{k}\cos ks, & k\in\mathbb{N}. \end{cases} \tag{4.3}$$

Proceeding as in the proof of Lemma 8.21 in [10] by some lengthy calculations for the integrals

$$\gamma_k(s) := \int_0^\pi \ln 2\,|\cos s - \cos\sigma|\sin k\sigma\, \mathrm{d}\sigma,$$

we can obtain the expressions

$$\gamma_k(s) = \frac{1}{k}\left\{[\cos ks - (-1)^k]\ln 4\cos^2\frac{s}{2} + [1-\cos ks]\ln 4\sin^2\frac{s}{2}\right.$$
$$\left. -2\sum_{j=1}^{k}\frac{\delta_{k-j}}{j}[1-(-1)^j]\cos(k-j)s\right\},\quad k\in\mathbb{N}. \tag{4.4}$$

Therefore, we arrive at the two trigonometric interpolatory quadratures

$$\int_0^\pi \ln 2|\cos s - \cos\sigma|\psi(\sigma)\, \mathrm{d}\sigma \approx \int_0^\pi \ln 2\,|\cos s - \cos\sigma|(P_{n,1}\psi)(\sigma)\, \mathrm{d}\sigma$$
$$= \sum_{j=0}^{n} R_j^{\mathrm{even}}(s)\psi(s_j) \tag{4.5}$$

for even functions ψ and

$$\int_0^\pi \ln 2|\cos s - \cos\sigma|\psi(\sigma)\, \mathrm{d}\sigma \approx \int_0^\pi \ln 2|\cos s - \cos\sigma|(P_{n,2}\psi)(\sigma)\, \mathrm{d}\sigma$$
$$= \sum_{j=1}^{n-1} R_j^{\mathrm{odd}}(s)\psi(s_j) \tag{4.6}$$

for odd functions ψ with the weights

$$R_j^{\mathrm{even}}(s) := -\frac{2\pi\delta_j}{n}\sum_{k=1}^{n}\frac{\delta_k}{k}\cos ks_j\cos ks$$

and

$$R_j^{\mathrm{odd}}(s) := \frac{2}{n}\sum_{k=1}^{n-1}\gamma_k(s)\sin ks_j.$$

Analogously, we have

$$\int_0^\pi \psi(\sigma)\, \mathrm{d}\sigma \approx \int_0^\pi (P_{n,1}\psi)(\sigma)\, \mathrm{d}\sigma = \frac{\pi}{n}\sum_{j=0}^{n}\delta_j\psi(s_j) \tag{4.7}$$

for even functions ψ and

$$\int_0^\pi \psi(\sigma)\,\mathrm{d}\sigma \approx \int_0^\pi (P_{n,2}\psi)(\sigma)\,\mathrm{d}\sigma = \sum_{j=1}^{n-1} w_j \psi(s_j) \tag{4.8}$$

for odd functions ψ with the weights

$$w_j := \frac{2}{n}\sum_{k=1}^{n-1}\frac{1}{k}\,[1-(-1)^k]\sin ks_j.$$

In view of (3.7) and the integrals

$$\frac{1}{2\pi}\int_0^{2\pi}\cot\frac{\sigma-s}{2}\cos k\sigma = -\sin ks, \quad k\in\mathbb{N},$$

we have that

$$(U\psi)(s) = \sum_{j=1}^{n-1} T_j^{\mathrm{odd}}(s)\psi(s_j) \tag{4.9}$$

for $\psi\in T_{n,2}$ with the weights given by

$$T_j^{\mathrm{odd}}(s) := -\frac{2}{n}\sum_{k=1}^{n-1} k\sin ks_j \sin ks.$$

After taking care of the logarithmic singularities of the kernels of the operators $A_{m\ell}$ by writing

$$(A_{m\ell}\psi)(s) = \int_0^\pi \{a_{m\ell}(s,\sigma)\ln 2|\cos s - \cos\sigma| + b_{m\ell}(s,\sigma)\}\psi(\sigma)\,\mathrm{d}\sigma$$

with infinitely differentiable functions $a_{m\ell}$ and $b_{m\ell}$ we now define approximating quadrature operators by

$$(A_{n,m\ell}\psi)(s) = \int_0^\pi \{\ln 2|\cos s - \cos\sigma|(P_{n,3-\ell}a_{m\ell}(s,\cdot)\psi)(\sigma) + (P_{n,3-\ell}b_{m\ell}(s,\cdot)\psi)(\sigma)\}\,\mathrm{d}\sigma$$

for $m,\ell = 1,2$. Here, we acknowledge the fact that due to the factor $\sin\sigma$ all the kernels are odd with respect to the variable σ. We note that, in view of the above quadrature formulas, by collocation we obtain

$$(A_{n,m,1}\psi)(s_j) = \sum_{k=1}^{n-1}\{R_k^{\mathrm{odd}}(s_j)a_{m,1}(s_j,s_k)\psi(s_k) + w_k b_{m,1}(s_j,s_k)\psi(s_k)\} \tag{4.10}$$

for even functions ψ and

$$(A_{n,m,2}\psi)(s_j) = \sum_{k=0}^{n}\left\{R_k^{\mathrm{even}}(s_j)a_{m,2}(s_j,s_k)\psi(s_k) + \frac{\pi\delta_k}{n}\,b_{m,2}(s_j,s_k)\psi(s_k)\right\} \tag{4.11}$$

for odd functions ψ and $m = 1,2$.

In operator notation, our fully discrete collocation method can be expressed by the approximating equation

$$\begin{pmatrix} -\psi_{n,1} \\ U\psi_{n,2} \end{pmatrix} + \begin{pmatrix} P_{n,1}A_{n,11} & P_{n,1}A_{n,12} \\ P_{n,2}A_{n,21} & P_{n,2}(A_{n,22}+B) \end{pmatrix} \begin{pmatrix} \psi_{n,1} \\ \psi_{n,2} \end{pmatrix} = \begin{pmatrix} P_{n,1}g_1 \\ P_{n,2}g_2 \end{pmatrix} \tag{4.12}$$

that we need to solve for $\psi_{n,1} \in T_{n,1}$ and $\psi_{n,2} \in T_{n,2}$. In the derivation of (4.12) we have used the fact that $P_{n,2}U\psi_n = U\psi_n$ for $\psi_n \in T_{n,2}$. Via collocation, the projected Eq. (4.12) is equivalent to solving

$$\begin{aligned} &[-\psi_{n,1} + A_{n,11}\psi_{n,1} + A_{n,12}\psi_{n,2}](s_j) = g_1(s_j), \quad j = 0,\ldots,n, \\ &[U\psi_{n,2} + A_{n,21}\psi_{n,1} + (A_{n,22}+B)\psi_{n,2}](s_j) = g_2(s_j), \quad j = 1,\ldots,n-1. \end{aligned} \tag{4.13}$$

In view of (4.9)–(4.11), clearly, (4.13) represents a $2n \times 2n$ linear system for the $2n$ nodal values $\psi_{n,1}(s_j)$, $j = 0,\ldots,n$, and $\psi_{n,2}(s_j)$, $j = 1,\ldots,n-1$.

Our convergence analysis is based on the estimate

$$\|P_{n,\ell}\psi - \psi\|_{0,\alpha} \leqslant c\,\frac{\ln n}{n^{\beta-\alpha}}\,\|\psi\|_{0,\beta}, \quad \ell = 1,2, \tag{4.14}$$

for the trigonometric interpolation which is valid for $0 < \alpha < \beta < 1$, and some constant c depending only on α and β (see [14, p. 78]). With the aid of Lemma 4.1 in [8] and (4.14) it can be seen that

$$\|(A_{n,m\ell} - A_{m\ell})\psi\|_{0,\alpha} \leqslant c_1\,\frac{\ln n}{n^{\beta-\gamma}}\,\|\psi\|_{0,\beta} \tag{4.15}$$

for $0 < \alpha,\ \beta < 1,\ 0 < \gamma < \beta$, and some constant c_1 depending on α, β, and γ. In particular, by the uniform boundedness principle, this implies that

$$\|A_{n,m\ell}\psi\|_{0,\alpha} \leqslant c_2\|\psi\|_{0,\beta}$$

for $0 < \alpha,\ \beta < 1$ and some constant c_2 depending on α and β. Therefore, we can estimate

$$\|(P_{n,\ell}A_{n,m\ell} - A_{n,m\ell})\psi\|_{0,\alpha} \leqslant c_3\,\frac{\ln n}{n^{\beta-\alpha}}\,\|A_{n,m\ell}\psi\|_{0,\beta} \leqslant c_2c_3\,\frac{\ln n}{n^{\beta-\alpha}}\,\|\psi\|_{0,\alpha} \tag{4.16}$$

for $0 < \alpha < \beta < 1$ and some constant c_3 depending on α and β. Summarizing from (4.15) and (4.16) we have that

$$\|(P_{n,\ell}A_{n,m\ell} - A_{m\ell})\psi\|_{0,\alpha} \leqslant C\,\frac{\ln n}{n^{\min\{\alpha,1-\alpha\}-\varepsilon}}\,\|\psi\|_{0,\alpha}, \tag{4.17}$$

for all $0 < \alpha < 1$, $0 < \varepsilon < \min\{\alpha, 1-\alpha\}$, and some constant C depending on α and ε. Together with the fact that the norm $\|\cdot\|_{1,\alpha}$ is stronger than $\|\cdot\|_{0,\beta}$ for all $0 < \alpha, \beta < 1$ from (4.14) and (4.17) we can conclude norm convergence

$$\begin{pmatrix} P_{n,1}A_{n,11} & P_{n,1}A_{n,12} \\ P_{n,2}A_{n,21} & P_{n,2}(A_{n,22}+B) \end{pmatrix} \to \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}+B \end{pmatrix}, \quad n \to \infty$$

with the operators again considered as mappings from $C^{0,\alpha}[0,\pi]\times C_0^{1,\alpha}[0,\pi]$ into $C^{0,\alpha}[0,\pi]\times C_0^{0,\alpha}[0,\pi]$. By the classical Neumann series argument this implies the following convergence theorem (see also [10, Corollary 13.13]).

Theorem 4.1. *For $g_1\in C^{0,\alpha}[0,\pi]$ and $g_2\in C_0^{0,\alpha}[0,\pi]$ and sufficiently large n the approximate equation* (4.12) *has a unique solution $\psi_{n,1}\in T_{n,1}$ and $\psi_{n,2}\in T_{n,2}$ and, for the exact solution ψ_1 and ψ_2 of* (3.5), *we have convergence*

$$\|\psi_{n,1}-\psi_1\|_{0,\alpha}+\|\psi_{n,2}-\psi_2\|_{1,\alpha},\quad n\to\infty.$$

We refrain from writing up the error estimate that results from the above convergence analysis, since due to our crude estimate (4.17) it would be to pessimistic with respect to the convergence order. An analysis leading to optimal estimates would have to incorporate the smoothing properties of the integral operators.

5. Numerical examples

In our numerical examples, we consider the scattering of plane waves given by

$$u^{\mathrm{i}}(x)=\mathrm{e}^{\mathrm{i}kd\cdot x},$$

where d denotes a unit vector giving the direction of propagation. For the scattered wave u^{s} we have to solve the impedance crack problem with boundary values

$$f_\pm=-\frac{\partial u^{\mathrm{i}}}{\partial\nu}\mp\mathrm{i}k\lambda u^{\mathrm{i}}\quad\text{on }\Gamma.$$

We illustrate the numerical results by the far field pattern u_∞ of the scattered wave that is defined by the asymptotic behavior of the scattered wave

$$u^{\mathrm{s}}(x)=\frac{\mathrm{e}^{\mathrm{i}k|x|}}{\sqrt{|x|}}\left\{u_\infty(\hat{x})+O\left(\frac{1}{|x|}\right)\right\},\quad |x|\to\infty,$$

uniformly for all directions $\hat{x}:=x/|x|$ (see [4]). From the asymptotics for the Hankel function for large argument, we see that the far-field pattern of the combined double- and single-layer potential (3.1) is given by

$$u_\infty(\hat{x})=\frac{\mathrm{e}^{-\mathrm{i}\pi/4}}{\sqrt{8\pi k}}\int_\Gamma\{\mathrm{i}\varphi_1(y)+k\hat{x}\cdot\nu(y)\varphi_2(y)\}\mathrm{e}^{-\mathrm{i}k\hat{x}\cdot y}\,\mathrm{d}s(y).\tag{5.1}$$

After parameterizing $y=z(\cos s)$ and inserting solution of the integral equation (3.5) via (3.6), integral (5.1) can be evaluated by quadratures (4.7) and (4.8).

Our first numerical results are for a straight crack with parametric representation

$$z(t)=(t,0),\quad -1\leqslant t\leqslant 1.$$

Table 1 gives the far field in the forward and backward scattering direction for the incident direction $d=(0,1)$ and constant impedance $\lambda=0.5$. A satisfactory convergence is exhibited.

Table 1
Results for a straight crack with constant impedance $\lambda = 0.5$

k	n	$\mathrm{Re}\, u_\infty(d)$	$\mathrm{Im}\, u_\infty(d)$	$\mathrm{Re}\, u_\infty(-d)$	$\mathrm{Im}\, u_\infty(-d)$
1	8	−0.29535834	0.77060173	−0.15403221	−0.45164349
	16	−0.29528352	0.77067030	−0.15405566	−0.45166171
	32	−0.29527906	0.77067456	−0.15405712	−0.45166289
	64	−0.29527879	0.77067483	−0.15405721	−0.45166296
3	8	−0.89924111	1.01393827	0.21991200	−0.37799311
	16	−0.89880085	1.01409351	0.22005825	−0.37795969
	32	−0.89878330	1.01410791	0.22005914	−0.37795392
	64	−0.89878223	1.01410881	0.22005919	−0.37795358

Table 2
Results for a straight crack with impedance (5.2)

k	n	$\mathrm{Re}\, u_\infty(d)$	$\mathrm{Im}\, u_\infty(d)$	$\mathrm{Re}\, u_\infty(-d)$	$\mathrm{Im}\, u_\infty(-d)$
1	8	−0.3303056536	0.6600859194	−0.1426504815	−0.3924396803
	16	−0.3302615280	0.6599618767	−0.1426195881	−0.3925606318
	32	−0.3302615265	0.6599618938	−0.1426195961	−0.3925606209
	64	−0.3302615265	0.6599618938	−0.1426195961	−0.3925606209
3	8	−0.7029933464	0.9685877542	0.1182326566	−0.4311620703
	16	−0.7033348762	0.9692304130	0.1184960325	−0.4316701385
	32	−0.7033341227	0.9692296704	0.1184967380	−0.4316708869
	64	−0.7033341226	0.9692296704	0.1184967379	−0.4316708869

Table 2 gives the corresponding results for the impedance

$$\lambda(t) = (1 - t^2)^2, \quad -1 \leqslant t \leqslant 1. \tag{5.2}$$

It shows a better convergence behavior since at the end points the impedance has a high order zero, i.e., at the end points we essentially have a Neumann boundary condition.

Fig. 1 show graphs of the modulus of the far-field pattern for the incident direction $d = (0, 1)$, the wave numbers $k = 1$ and 3 and the constant impedances $\lambda = 0.1$, 1 and 10. The star indicates the location of the origin.

To illustrate that our method also works satisfactory for nonstraight cracks, as a final example we consider scattering from a semi-circular crack with parametric representation

$$z(t) = \left(\cos\frac{\pi}{2}\,t, \sin\frac{\pi}{2}\,t\right), \quad -1 \leqslant t \leqslant 1.$$

Table 3 shows the numerical results for the incident direction $d = (1, 0)$ and a constant impedance $\lambda = 0.5$.

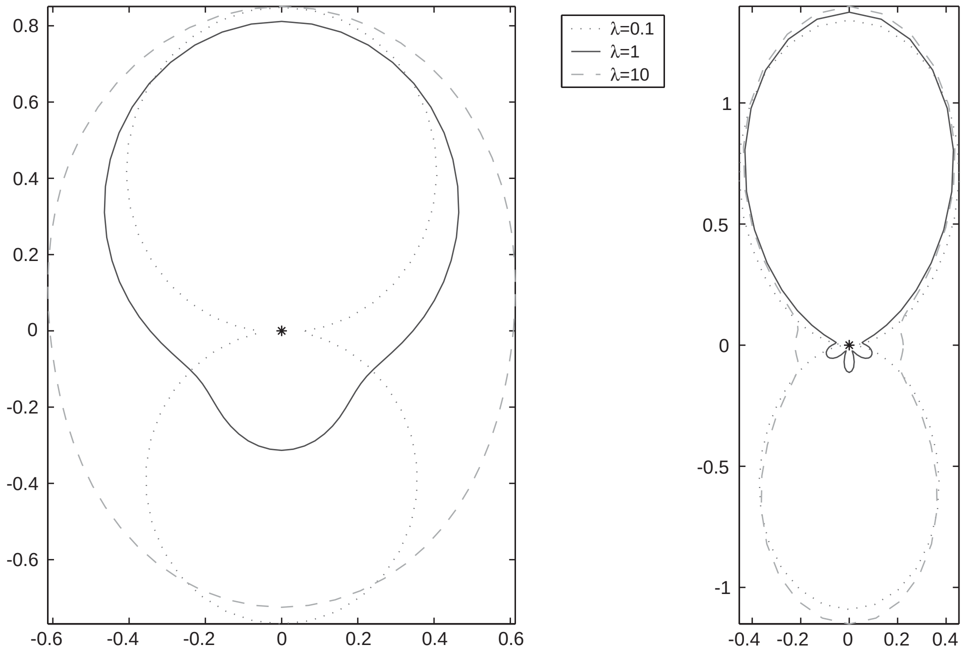

Fig. 1. Modulus of far field for $k = 1$ (left) and $k = 3$ (right).

Table 3
Results for a semi-circular crack with constant impedance $\lambda = 0.5$

k	n	$\mathrm{Re}\, u_\infty(d)$	$\mathrm{Im}\, u_\infty(d)$	$\mathrm{Re}\, u_\infty(-d)$	$\mathrm{Im}\, u_\infty(-d)$
1	8	−0.2380129767	0.2070162491	−0.1368373850	0.0498794864
	16	−0.2380022863	0.2070143584	−0.1368393889	0.0498866325
	32	−0.2380016513	0.2070142737	−0.1368395463	0.0498870796
	64	−0.2380016121	0.2070142690	−0.1368395564	0.0498871072
3	8	−0.4107084858	0.3220779742	0.0136662347	0.0154637519
	16	−0.4105663749	0.3220662977	0.0137804260	0.0149074978
	32	−0.4105617439	0.3220674704	0.0137800020	0.0149066760
	64	−0.4105614615	0.3220675491	0.0137799815	0.0149066262

References

[1] F. Cakoni, D. Colton, The linear sampling method for cracks, Inverse Problems 19 (2003) 279–295.
[2] M.R. Capobianco, G. Criscuolo, P. Junghanns, A fast algorithm for Prandtl's integro-differential equation, J. Comput. Appl. Math. 77 (1997) 103–128.
[3] R. Chapko, R. Kress, L. Mönch, On the numerical solution of a hypersingular integral equation for elastic scattering from a planar crack, IMA J. Numer. Anal. 20 (2000) 601–619.
[4] D. Colton, K. Kress, Inverse Acoustic and Electromagnetic Scattering Theory, 2nd Edition, Springer, Berlin, 1998.
[5] M. Costabel, M. Dauge, Crack singularities for general elliptic systems, Math. Nachr. 235 (2002) 29–49.

[6] R. Kress, A Nyström method for boundary integral equations in domains with corners, Numer. Math. 58 (1990) 145–161.
[7] R. Kress, Inverse scattering from an open arc, Math. Methods Appl. Sci. 18 (1995) 267–293.
[8] R. Kress, On the numerical solution of a hypersingular integral equation in scattering theory, J. Comput. Appl. Math. 61 (1995) 345–360.
[9] R. Kress, Inverse elastic scattering from a crack, Inverse Problems 12 (1996) 667–684.
[10] R. Kress, Linear Integral Equations, 2nd Edition, Springer, New York, 1999.
[11] L. Mönch, On the numerical solution of the direct scattering problem for a sound-hard open arc, J. Comput. Appl. Math. 71 (1996) 343–356.
[12] H. Multhopp, Die Berechnung der Auftriebsverteilung von Tragflügeln, Luftfahrt-Forschung 4 (1938) 153–169.
[13] A.V. Osipov, A.N. Norris, The Malyuzhinets theory for scattering from wedge boundaries: a review, Wave Motion 29 (1999) 313–340.
[14] S. Prössdorf, B. Silbermann, Numerical Analysis for Integral and Related Operator Equations, Akademie-Verlag, Berlin, Birkhäuser-Verlag, Basel, 1991.
[15] Y. Yan, I.H. Sloan, On integral equations of the first kind with logarithmic kernels, J. Integral Equations Appl. 1 (1988) 549–579.